

Construction of R^4 Terms in N=2 D=8 Superspace

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Using linearized superfields, R^4 terms in the Type II superstring effective action compactified on T^2 are constructed as integrals in N=2 D=8 superspace. The structure of these superspace integrals allows a simple proof of the R^4 non-renormalization theorems which were first conjectured by Green and Gutperle.

1. Introduction

One way to verify that M-theory provides a consistent non-perturbative definition of superstring theory is to compare M-theory predictions with superstring scattering amplitudes. In general, this comparison is difficult since one needs to know the non-perturbative behavior of the superstring amplitudes.

However, one special case where comparison is possible is the R^4 term in the Type II superstring effective action. For the Type IIB superstring, this term only receives tree-level, one-loop, and non-perturbative corrections, while for the Type IIA superstring, it only receives tree-level and one-loop contributions. This loop-dependence of R^4 terms was first conjectured on the basis of $SL(2,Z)$ symmetry[1] and was later verified by comparison with R^2 terms in the effective action obtained by compactification on $K3 \times T^2$ [2].

The loop-dependence of R^4 terms is reminiscent of terms in the effective action of the Type II superstring compactified to four dimensions, where low-energy decoupling of $N=2$ $D=4$ vector and hypermultiplets implies various non-renormalization theorems.[3] The easiest way to prove this decoupling is to note that low-energy terms come from $N=2$ $D=4$ superspace actions which only integrate over half of the eight θ 's. Since vector multiplets are represented by chiral superfields and hypermultiplets (or more accurately, tensor hypermultiplets) are represented by linear superfields, there is no local superspace action containing both types of superfields which only depends on half the θ 's[4].

As will be shown in this paper, a similar argument can be used to prove non-renormalization theorems for R^4 terms. This is done by first constructing $N=2$ $D=8$ superspace integrals for R^4 terms in the effective action of the Type II superstring compactified on T^2 . Although there is only one irreducible $N=2$ $D=8$ multiplet[5], it can be represented at linearized level as either a chiral superfield (whose lowest component is a complex scalar parameterizing $SL(2)/SO(2)$) or as a linear superfield (whose lowest components are five scalars parameterizing $SL(3)/SO(3)$).

As in the $N=2$ $D=4$ case, the chiral and linear superfield can not both appear in $N=2$ $D=8$ superspace actions which integrate over half of the thirty-two θ 's. This implies that the zero modes of the $SL(2)/SO(2)$ scalars and $SL(3)/SO(3)$ scalars decouple. After constructing R^4 terms as $N=2$ $D=8$ superspace integrals over sixteen θ 's, this decoupling is used to prove that R^4 terms in the effective action of the uncompactified Type IIB superstring only appear at tree-level, one-loop and non-perturbatively, while R^4 terms in the effective action of the uncompactified Type IIA superstring only appear at tree-level and one-loop.

In the second section of this paper, the proof of non-renormalization theorems for $N=2$ $D=4$ systems is reviewed. In the third section, a chiral and linear superfield is constructed for the $N=2$ $D=8$ supergravity multiplet at linearized level and $N=2$ $D=8$ superspace actions are constructed for terms with eight derivatives. In the fourth section, this construction is used to prove non-renormalization theorems for R^4 terms in the effective action of the uncompactified Type IIA and Type IIB superstrings. In the fifth section, possible generalizations of the R^4 non-renormalization theorems are discussed.

2. Review of the $N=2$ $D=4$ Action for the Compactified Type II Superstring

Since the superspace structure of the $N=2$ $D=8$ effective action closely resembles that of the $N=2$ $D=4$ effective action, the $N=2$ $D=4$ superspace effective action[4] for the compactified Type II superstring will be reviewed first.

2.1. *Scalar versus Tensor Hypermultiplets*

The matter superfields present in the $N=2$ $D=4$ action consist of vector multiplets and hypermultiplets. In four dimensions, tensors are on-shell equivalent to scalars, so one can either represent the hypermultiplets by tensor hypermultiplets (containing one tensor and three scalars) or by scalar hypermultiplets (containing four scalars). Although most of the literature uses scalar hypermultiplets, $N=2$ $D=4$ superspace effective actions are much easier to construct if one uses tensor hypermultiplets.

Furthermore, superstring field theory and sigma model arguments suggest that the correct off-shell representation is the tensor hypermultiplet rather than the scalar hypermultiplet. As will be discussed in the following subsection, $N=2$ $D=4$ vector and tensor multiplets are described by chiral/chiral and chiral/anti-chiral spacetime superfields. This structure is predicted by string field theory[6] since a $D=4$ Type II closed superstring field should be the product of two $D=4$ open superstring fields, and the massless sector of the open $D=4$ superstring includes $N=1$ $D=4$ Wess-Zumino scalar multiplets which are described by chiral and anti-chiral superfields.

This same superspace structure is also predicted by the spacetime-supersymmetric sigma model for the $D=4$ Type II superstring, which relates superspace chirality and worldsheet chirality[7][4]. For the Type IIB (Type IIA) superstring, the chiral/chiral spacetime superfields for vector multiplets come from chiral/chiral (chiral/anti-chiral) worldsheet moduli of the Calabi-Yau manifold and the chiral/anti-chiral spacetime superfields for the

tensor hypermultiplets come from chiral/anti-chiral (chiral/chiral) worldsheet moduli of the Calabi-Yau manifold.

At least at the perturbative level, representing some Ramond-Ramond scalars by tensors does not restrict the action because the R-R zero modes decouple, so one can obtain the scalar action by performing a duality transformation on the tensor action. At the non-perturbative level, there could be difficulties when the abelian gauge-invariance of the tensor is broken to a discrete subgroup in the effective action.[8] However, as discussed in [9], it appears possible to describe even these actions with tensor multiplets.

2.2. $N=2$ $D=4$ Superfields

The variables of $N=2$ $D=4$ superspace are $[x^\mu, \theta_j^\alpha, \bar{\theta}^{j\dot{\alpha}}]$ where $\mu = 0$ to 3, α and $\dot{\alpha} = 1$ to 2, and $j = 1$ to 2 is an internal $SU(2)_R$ index which is raised and lowered using the anti-symmetric ϵ^{jk} tensor. $\bar{\theta}^{j\dot{\alpha}}$ is the complex conjugate of θ_j^α , and under $U(1)_R$ transformations, θ_j^α carries +1 charge and $\bar{\theta}^{j\dot{\alpha}}$ carries -1 charge.

Under supersymmetry transformations parameterized by ξ_j^α and $\bar{\xi}_j^{\dot{\alpha}}$,

$$\delta\theta_j^\alpha = \xi_j^\alpha, \quad \delta\bar{\theta}^{j\dot{\alpha}} = \bar{\xi}^{j\dot{\alpha}}, \quad \delta x^\mu = i\sigma_{\alpha\dot{\alpha}}^\mu(\xi_j^\alpha\bar{\theta}^{j\dot{\alpha}} + \bar{\xi}^{j\dot{\alpha}}\theta_j^\alpha),$$

and supersymmetric derivatives are defined as

$$D_\alpha^j = \frac{\partial}{\partial\theta_j^\alpha} + i\bar{\theta}^{j\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu, \quad \bar{D}_{\dot{\alpha}}^j = \frac{\partial}{\partial\bar{\theta}_j^{\dot{\alpha}}} + i\theta_{j\alpha}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\partial^\mu. \quad (2.1)$$

The field-strength of a vector multiplet is described by a restricted chiral superfield W satisfying

$$\begin{aligned} \bar{D}_{\dot{\alpha}}^+ W &= \bar{D}_{\dot{\alpha}}^- W = 0, \\ D_\alpha^{(j} D^{k)\alpha} W &= \bar{D}_{\dot{\alpha}}^{(j} D^{k)\dot{\alpha}} \bar{W} \end{aligned} \quad (2.2)$$

where the first constraint implies that W is chiral/chiral, while the second constraint implies that W is restricted. The physical bosonic components of W appear as

$$W = w(x) + \theta_j^{(\alpha}\theta^{\beta)j}\sigma_{\alpha\beta}^{\mu\nu}F_{\mu\nu}(x) + \dots \quad (2.3)$$

where w is a complex scalar and $F_{\mu\nu}$ is the vector field strength. Under $U(1)_R \times SU(2)_R$, W transforms as $(+2, 1)$, so w and \bar{w} transform as $(+2, 1)$ and $(-2, 1)$ while $F_{\mu\nu}$ transforms as $(0, 1)$.

The field-strength of a tensor hypermultiplet is described by a linear superfield L_{jk} symmetric in its $SU(2)$ indices which satisfies the reality condition $L_{jk} = (L^{jk})^*$ and the linear constraint

$$D_{(j}^\alpha L_{kl)} = 0, \quad \bar{D}_{(j}^{\dot{\alpha}} L_{kl)} = 0. \quad (2.4)$$

The physical bosonic components of L_{jk} appear as

$$L_{jk} = l_{jk}(x) + \theta_{(j}^\alpha \bar{\theta}_{k)}^{\dot{\alpha}} \epsilon_{\mu\nu\rho\kappa} \sigma_{\alpha\dot{\alpha}}^\mu H^{\nu\rho\kappa}(x) + \dots \quad (2.5)$$

where l_{jk} is a triplet of scalars transforming as $(0, 3)$ under $U(1)_R \times SU(2)_R$ and $H^{\mu\nu\rho}$ is the tensor field-strength which transforms as $(0, 1)$.

Although the constraints of (2.4) appear very different from the constraints of (2.2), they are actually closely related. This can be seen by noting that the constraints of (2.4) imply that L_{++} is restricted twisted-chiral since it satisfies

$$D_+^\alpha L_{++} = \bar{D}^{-\dot{\alpha}} L_{++} = 0, \quad (2.6)$$

$$D_-^\alpha D_{-\alpha} L_{++} = D_+^\alpha D_{+\alpha} L_{--}, \quad \bar{D}_-^{\dot{\alpha}} \bar{D}_{-\dot{\alpha}} L_{++} = \bar{D}_+^{\dot{\alpha}} \bar{D}_{+\dot{\alpha}} L_{--}.$$

The first two constraints imply that L_{++} is chiral/anti-chiral, while the second two constraints imply that L_{++} is restricted.

2.3. $N=2$ $D=4$ Superspace Actions

Two-derivative actions for M vector multiplets and N tensor hypermultiplets can be written in manifestly supersymmetric notation as

$$\int d^4x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} [(D_+)^2 (D_-)^2 f_V(W^{(I)}) + \oint_0 d\zeta (D_-)^2 (\bar{D}^+)^2 f_T(\tilde{L}^{(J)}, \zeta) + c.c.] \quad (2.7)$$

where $I = 1$ to M , $J = 1$ to N , f_V and f_T are arbitrary functions of M and $N + 1$ variables, $\oint_0 d\zeta$ is a contour integration around $\zeta = 0$, and

$$\tilde{L}^{(J)} = L_{++}^{(J)} + \zeta L_{+-}^{(J)} + \zeta^2 L_{--}^{(J)}. \quad (2.8)$$

The hypermultiplet contribution to (2.7) is supersymmetric where

$$\delta_Q f_T = [\xi_j^\alpha D_\alpha^j + \bar{\xi}_{\dot{\alpha}}^j \bar{D}_j^{\dot{\alpha}} - 2i(\xi \sigma^\mu \bar{\theta} + \bar{\xi} \bar{\sigma}^\mu \theta) \partial_\mu] f_T, \quad (2.9)$$

since $D_+^\alpha f_T = \zeta D_-^\alpha f_T$ and $\bar{D}_-^{\dot{\alpha}} f_T = \zeta \bar{D}_+^{\dot{\alpha}} f_T$, so $(D_-)^2 (\bar{D}^+)^2 \delta_Q f_T$ is a total derivative in x^μ [10].

Note that integrating over all eight θ 's (i.e. using $(D_+)^2(D_-)^2(\bar{D}^+)^2(\bar{D}^-)^2$) would imply a minimum of four derivatives in the action, assuming the action is local. In this paper, non-local actions (such as those coming from holomorphic anomalies[11][12] which involve $(\partial^\mu\partial_\mu)^{-1}$) will not be discussed.

As shown in [4], the above action can be easily coupled to supergravity by introducing a compensating vector multiplet, a compensating tensor hypermultiplet, and a physical tensor hypermultiplet which is the ‘universal hypermultiplet’.

2.4. $N=2$ $D=4$ Non-Renormalization Theorems

To prove non-renormalization theorems, one uses the fact that the zero modes of Ramond-Ramond fields decouple at the perturbative level. Ramond-Ramond zero modes only appear in the lowest components of $L_{+-}^{(J)}$, so the perturbative contribution to (2.7) needs to be invariant under $\tilde{L}^{(J)} \rightarrow \tilde{L}^{(J)} + \zeta C^{(J)}$ where $C^{(J)}$ are constants. This is true whenever $f_T(\tilde{L}^{(J)}, \zeta) = \zeta^{-1} g(\tilde{L}^{(J)})$ where $g(\tilde{L}^{(J)})$ is a function of N variables since, in this case, the shift in $\tilde{L}^{(J)}$ cancels the pole when $\zeta = 0$.

So at the perturbative level, the hypermultiplet contribution to the action of (2.7) simplifies to

$$\begin{aligned} & \int d^4x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta \zeta^{-1} (D_-)^2 (\bar{D}^+)^2 g(\tilde{L}^{(J)}) + c.c. \\ &= \int d^4x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} (D_-)^2 (\bar{D}^+)^2 g(L_{++}^{(J)}) + c.c. \end{aligned} \quad (2.10)$$

It is interesting to note that if $f_V(W^{(I)})$ is the Type IIA vector potential on some Calabi-Yau manifold, then $g(L_{++}^{(I)}) = f_V(L_{++}^{(I)})$ is the Type IIB hypermultiplet potential on the same Calabi-Yau manifold.¹

As shown in [4] using two-dimensional sigma model arguments, the vector and perturbative hypermultiplet contribution to (2.7) scales like c^{-2} as one scales the string coupling-constant $\lambda_s \rightarrow c\lambda_s$, implying that they only contribute at tree-level. However, the hypermultiplet contribution to the action could also get non-perturbative contributions which depend on Ramond-Ramond zero modes, e.g.

$$\int d^4x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta (D_-)^2 (\bar{D}^+)^2 (e^{-\tilde{L}^{(1)}/\zeta} f_T(\tilde{L}^{(J)}, \zeta)) + c.c. \quad (2.11)$$

¹ This relation can be proven using sigma model arguments[4]. However, it does not follow directly from mirror symmetry as was mistakenly claimed in [4].

3. Structure of N=2 D=8 Superspace

The structure of superfields and actions in N=2 D=8 superspace is almost identical to the structure in N=2 D=4 superspace, at least for on-shell linearized superfields. This will allow the methods of the previous section to be repeated in this section.

3.1. N=2 D=8 Superfields

The variables of N=2 D=8 superspace are $[x^\mu, \theta_j^\alpha, \bar{\theta}^{j\dot{\alpha}}]$ where $\mu = 0$ to 7, α and $\dot{\alpha} = 1$ to 8, and $j = 1$ to 2 is an internal $SU(2)_R$ index which is raised and lowered using the anti-symmetric ϵ^{jk} tensor. $\bar{\theta}^{j\dot{\alpha}}$ is the complex conjugate of θ_j^α , and under $U(1)_R$ transformations, θ_j^α carries +1 charge and $\bar{\theta}^{j\dot{\alpha}}$ carries -1 charge.

Under supersymmetry transformations parameterized by ξ_j^α and $\bar{\xi}_j^{\dot{\alpha}}$,

$$\delta\theta_j^\alpha = \xi_j^\alpha, \quad \delta\bar{\theta}^{j\dot{\alpha}} = \bar{\xi}_j^{\dot{\alpha}}, \quad \delta x^\mu = i\sigma_{\alpha\dot{\alpha}}^\mu(\xi_j^\alpha\bar{\theta}^{j\dot{\alpha}} + \bar{\xi}_j^{\dot{\alpha}}\theta_j^\alpha),$$

where $\sigma_{\alpha\dot{\alpha}}^\mu$ is the standard SO(8) Pauli matrix, and supersymmetric derivatives are defined as

$$D_\alpha^j = \frac{\partial}{\partial\theta_j^\alpha} + i\bar{\theta}^{j\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu, \quad \bar{D}_{\dot{\alpha}}^j = \frac{\partial}{\partial\bar{\theta}_j^{\dot{\alpha}}} + i\theta_{j\alpha}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\partial^\mu. \quad (3.1)$$

The massless sector of the Type II superstring compactified on T^2 is an N=2 D=8 supergravity multiplet[5] whose 128 bosonic fields include 7 scalars, 6 vectors, 3 anti-symmetric two-forms, one anti-symmetric three-form, and a spin-two graviton. At linearized level, these fields can be combined on-shell into a chiral superfield W and a linear superfield L_{jklm} which is symmetric in its SU(2) indices and satisfies the reality condition $L_{jklm} = (L^{jklm})^*$.

In addition to the chiral and linear constraints

$$\bar{D}_{\dot{\alpha}}^j W = D_\alpha^j \bar{W} = 0, \quad D_{(j}^\alpha L_{klmn)} = \bar{D}_{(\dot{j}}^{\dot{\alpha}} L_{klmn)} = 0, \quad (3.2)$$

these superfields are related to each other by the constraint

$$D_j^\alpha D_k^\beta \sigma_{\alpha\beta}^{\mu\nu} W = \bar{D}^{l\dot{\alpha}} \bar{D}^{m\dot{\beta}} \bar{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} L_{jklm}, \quad \bar{D}_j^{\dot{\alpha}} \bar{D}_k^{\dot{\beta}} \bar{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} \bar{W} = D^{l\alpha} D^{m\beta} \sigma_{\alpha\beta}^{\mu\nu} L_{jklm}. \quad (3.3)$$

Note that L_{++++} is a chiral/anti-chiral field satisfying $\bar{D}_{\dot{\alpha}}^- L_{++++} = D_+^\alpha L_{++++} = 0$ and

$$(D_- \sigma^{\mu\nu} D_-) L_{++++} = (\bar{D}^- \sigma^{\mu\nu} \bar{D}^-) \bar{W}, \quad (\bar{D}^+ \bar{\sigma}^{\mu\nu} \bar{D}^+) L_{++++} = (D_+ \sigma^{\mu\nu} D_+) W. \quad (3.4)$$

The physical bosonic fields appear in W and L_{jklm} as

$$W = w + (\theta_j \sigma^{\mu\nu} \theta_k) F_{\mu\nu}^{jk} + (\theta_j \sigma^{\mu\nu\rho\kappa} \theta^j) F_{\mu\nu\rho\kappa} + (\theta_j \sigma^{\mu\nu} \theta_k) (\theta^j \sigma^{\rho\kappa} \theta^k) R_{\mu\nu\rho\kappa} + \dots, \quad (3.5)$$

$$\begin{aligned} L_{jklm} = & l_{jklm} + (\theta_j \sigma_{\mu\nu} \theta_k) F_{lm}^{\mu\nu} + (\bar{\theta}_j \bar{\sigma}_{\mu\nu} \bar{\theta}_k) \bar{F}_{lm}^{\mu\nu} + (\theta_j \sigma_{\mu\nu\rho} \bar{\theta}_k) H_{lm}^{\mu\nu\rho} \\ & + (\theta_j \sigma^{\mu\nu} \theta_k) (\bar{\theta}_l \bar{\sigma}^{\rho\kappa} \bar{\theta}_m) R_{\mu\nu\rho\kappa} + \dots \end{aligned}$$

Under $U(1)_R \times SU(2)_R$, W transforms as $(+4, 1)$ and L_{jklm} transforms as $(0, 5)$, so the scalars w and \bar{w} transform as $(+4, 1)$ and $(-4, 1)$, the scalars l_{jklm} transforms as $(0, 5)$, the vector field-strengths $F_{jk}^{\mu\nu}$ and $\bar{F}_{jk}^{\mu\nu}$ transform as $(+2, 3)$ and $(-2, 3)$, the tensor field-strength $H_{jk}^{\mu\nu\rho}$ transforms as $(0, 3)$, the self-dual and anti-self-dual part of the four-form field-strength $F^{\mu\nu\rho\kappa}$ transform as $(+1, 0)$ and $(-1, 0)$, and the curvature tensor $R_{\mu\nu\rho\kappa}$ transforms as $(0, 1)$.

The easiest way to understand the constraints of (3.3) is to use the fact that the Type II closed superstring field should be the ‘product’ of two open superstring fields.[6] The massless sector of the open superstring on T^2 is an N=1 D=8 super-Maxwell multiplet whose on-shell fields are described by a chiral and anti-chiral superfield, Φ and $\bar{\Phi}$, which satisfy the constraint

$$(D\sigma^{\mu\nu}D)\Phi = (\bar{D}\bar{\sigma}^{\mu\nu}\bar{D})\bar{\Phi}. \quad (3.6)$$

(In light-cone gauge, (3.6) is the familiar self-duality constraint[13] $D^a D^b \Phi = \epsilon^{abcd} \bar{D}_c \bar{D}_d \bar{\Phi}$ where θ^a is an SU(4) spinor.)

So it is natural to interpret the chiral/anti-chiral superfield W as the product $\Phi_L \Phi_R$ and the chiral/anti-chiral superfield L_{++++} as the product $\Phi_L \bar{\Phi}_R$, where Φ_L and Φ_R are ‘left-moving’ and ‘right-moving’ open string fields, and $(\theta_j^\alpha, \bar{\theta}_\alpha^j)$ split into left-moving $(\theta_-^\alpha, \bar{\theta}_\alpha^-)$ and right-moving $(\theta_+^\alpha, \bar{\theta}_\alpha^+)$. The constraint of (3.6), when applied independently on Φ_L and Φ_R , implies the constraints of (3.4),

$$(D_- \sigma^{\mu\nu} D_-) L_{++++} = (\bar{D}^- \sigma^{\mu\nu} \bar{D}^-) \bar{W}, \quad (\bar{D}^+ \bar{\sigma}^{\mu\nu} \bar{D}^+) L_{++++} = (D_+ \sigma^{\mu\nu} D_+) W. \quad (3.7)$$

3.2. N=2 D=8 Superspace Actions

Although the constraint of (3.3) puts W and L_{jklm} on-shell, one can ask what kinds of supersymmetric actions can be constructed out of the on-shell N=2 D=8 supergravity fields. This analysis will be useful for proving non-renormalization theorems for the on-shell S-matrix.

Eight-derivative actions can be written in manifestly N=2 D=8 supersymmetric notation as

$$\int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0}[(D_+)^8(D_-)^8f_V(W) + \oint_0 d\zeta(D_-)^8(\bar{D}^+)^8f_T(\tilde{L},\zeta) + c.c.] \quad (3.8)$$

where f_V and f_T are arbitrary functions, $\oint_0 d\zeta$ is a contour integration around $\zeta = 0$, and

$$\tilde{L} = L_{++++} + \zeta L_{+++-} + \zeta^2 L_{+-+--} + \zeta^3 L_{+----} + \zeta^4 L_{-----}. \quad (3.9)$$

The action is supersymmetric for the same reason as (2.7). Note that integrating over all 32 θ 's (i.e. using $(D_+)^8(D_-)^8(\bar{D}^+)^8(\bar{D}^-)^8$) would imply a minimum of sixteen derivatives in the action, assuming the action is local. As in the N=2 D=4 case, non-local actions such as those coming from holomorphic anomalies will not be discussed.

Although (3.8) is written in terms of linearized superfields, note that the U-duality group is $SL(3) \times SL(2)/SO(3) \times SO(2)$, and the internal automorphism group of the supersymmetry algebra is $SO(3)_R \times SO(2)_R$. This suggests that the full non-linear formulation of N=2 D=8 supergravity should include compensating scalars which parameterize $SO(2) \times SO(3)$, just like the scalars in the vector and tensor compensators of the N=2 D=4 superstring effective action[4].

4. Non-Renormalization Theorems for R^4 Terms

The first step in proving non-renormalization theorems is to use the fact that Ramond-Ramond zero modes decouple from perturbative amplitudes. As will be shown later, the only Ramond-Ramond zero modes appearing in W and L_{jklm} are the lowest components of L_{++++} and L_{----+} . Therefore, the perturbative contribution to the action of (3.8) must be invariant under $\tilde{L} \rightarrow \tilde{L} + C\zeta + \bar{C}\zeta^3$ where c is a complex constant.

When $f_T(\tilde{L},\zeta) = \zeta^{-1}g(\tilde{L})$ for an arbitrary function g , this is satisfied since the pole at $\zeta = 0$ is cancelled by the variation of \tilde{L} . This type of term contributes

$$\begin{aligned} & \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta \zeta^{-1} (D_-)^8 (\bar{D}^+)^8 g(\tilde{L}) + c.c. \\ &= \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} (D_-)^8 (\bar{D}^+)^8 g(L_{++++}) + c.c. \end{aligned}$$

to the action of (3.8).

However, unlike the N=2 D=4 case, there is another type of perturbative contribution from L_{jklm} which is given by $f_T(\tilde{L}, \zeta) = h\zeta^{-3}\tilde{L}^5$ where h is a constant. Under $\tilde{L} \rightarrow \tilde{L} + C\zeta + \bar{C}\zeta^3$, the variation proportional to \bar{C} cancels the pole when $\zeta = 0$. Furthermore, the term proportional to C does not contribute since

$$\begin{aligned} & \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta \zeta^{-2} (D_-)^8 (\bar{D}^+)^8 \tilde{L}^4 + c.c. \\ &= \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta \zeta^{-2} (\zeta^{-1} D_+)^8 (\zeta^{-1} \bar{D}^-)^8 \tilde{L}^4 + c.c. \\ &= \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} \oint_0 d\zeta \zeta^{-18} (D_+)^8 (\bar{D}^-)^8 \tilde{L}^4 + c.c., \end{aligned}$$

which vanishes since \tilde{L}^4 has a maximum of 16 ζ 's, so there is no term proportional to ζ^{-1} . It is easy to check that $f_T = \zeta^{-3}\tilde{L}^5$ is the only non-trivial term of this type (e.g. when $f_T = \zeta^{-3}\tilde{L}^4$, the action vanishes identically).

So at the perturbative level, the only possible terms in the action are

$$\begin{aligned} & \int d^8x|_{\theta_j^\alpha=\bar{\theta}_j^{\dot{\alpha}}=0} [(D_+)^8 (D_-)^8 f_V(W) + \\ & (D_-)^8 (\bar{D}^+)^8 (g(L_{++++}) + 5h(L_{++++}^4 L_{++--} + 2L_{++++}^3 L_{+++-}^2))] + c.c. . \end{aligned} \quad (4.1)$$

In components, it is easy to compute that this gives

$$\begin{aligned} & \int d^8x [R_{++}^4 (\frac{\partial}{\partial w})^4 f_V(w) + \\ & R_{+-}^4 \left((\frac{\partial}{\partial l_{++++}})^4 g(l_{++++}) + 120 h l_{++--} \right) + c.c.] + \dots \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} R_{\pm\pm}^4 &= (t^8 \pm \frac{i}{2}\epsilon^8)^{\mu_1\nu_1\ldots\mu_4\nu_4} (t^8 \pm \frac{i}{2}\epsilon^8)^{\rho_1\kappa_1\ldots\rho_4\kappa_4} \Pi_{n=1}^4 R^{\mu_n\nu_n\rho_n\kappa_n}, \\ (t^8 + \frac{i}{2}\epsilon^8)^{\mu_1\nu_1\ldots\mu_4\nu_4} &= \epsilon^{\alpha_1\beta_1\ldots\alpha_4\beta_4} \Pi_{n=1}^4 \sigma_{\alpha_n\beta_n}^{\mu_n\nu_n}, \\ (t^8 - \frac{i}{2}\epsilon^8)^{\mu_1\nu_1\ldots\mu_4\nu_4} &= \epsilon^{\dot{\alpha}_1\dot{\beta}_1\ldots\dot{\alpha}_4\dot{\beta}_4} \Pi_{n=1}^4 \bar{\sigma}_{\dot{\alpha}_n\dot{\beta}_n}^{\mu_n\nu_n}, \end{aligned} \quad (4.3)$$

and ... contains no R^4 terms. Note that $(t^8 \pm \frac{i}{2}\epsilon^8)$ is defined here with a D=8 Minkowski signature.

At the non-perturbative level, Ramond-Ramond zero modes do not have to decouple so one can have a term like

$$\int d^8x [R_{+-}^4 e^{-l_{++--}} (\frac{\partial}{\partial l_{++++}})^4 f(l_{++++}) + c.c.] \quad (4.4)$$

which comes from the superspace expression

$$\int d^8x |_{\theta_j^\alpha = \bar{\theta}_j^{\dot{\alpha}} = 0} \oint_0 d\zeta \zeta^{-1} (D_-)^8 (\bar{D}^+)^8 \left(f(\tilde{L}) e^{-\tilde{L}/\zeta^2} \right) + c.c. . \quad (4.5)$$

To determine the loop-order of the terms in (4.1) and to determine which terms survive in the uncompactified limit, one needs to know how the scalar fields depend on the string coupling constant and on the T^2 volume. Although the scalars l_{jklm} and w are only defined at linearized level, this will be enough to prove non-renormalization theorems for the R^4 terms.

The seven scalar moduli consist of the complex modulus $U = U_1 + iU_2$ of the two-torus, the kahler modulus $T = T_1 + iT_2$ of the two-torus (T_2 is the volume), the complex Ramond-Ramond scalar $B = B_1 + iB_2$, and the D=8 string coupling constant λ_8 which is related to the D=10 string coupling constant by $\lambda_8 = (T_2)^{-\frac{1}{2}} \lambda_{10}$.

These moduli for the Type IIB superstring can be combined into the following symmetric matrices with determinant one[14]

$$M_1 = \frac{1}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \quad M_2 = \frac{1}{(\lambda_8)^{2/3} T_2} \begin{pmatrix} 1 & T_1 & -B_1 \\ T_1 & |T|^2 & \text{Re}(\bar{T}B) \\ -B_1 & \text{Re}(\bar{T}B) & T_2(\lambda_8)^2 + |B|^2 \end{pmatrix} \quad (4.6)$$

which transform as $M_a \rightarrow \Omega_a M_a \Omega_a^T$ under the $SL(2, \mathbb{R})$ and $SL(3, \mathbb{R})$ transformations generated by Ω_1 and Ω_2 . For the Type IIA superstring, the only difference is that the T and U moduli switch places.

Expanding to first order near $M_1 = M_2 = 1$ (i.e. $T = U = i$, $\lambda_8 = 1$, $B = 0$), one finds

$$M_1 = \begin{pmatrix} 1 - \hat{U}_2 & \hat{U}_1 \\ \hat{U}_1 & 1 + \hat{U}_2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 - \hat{T}_2 - \frac{2}{3} \hat{\lambda}_8 & \hat{T}_1 & -\hat{B}_1 \\ \hat{T}_1 & 1 + \hat{T}_2 - \frac{2}{3} \hat{\lambda}_8 & \hat{B}_2 \\ -\hat{B}_1 & \hat{B}_2 & 1 + \frac{4}{3} \hat{\lambda}_8 \end{pmatrix} \quad (4.7)$$

where $\hat{U} = U - i$, $\hat{T} = T - i$, $\hat{\lambda}_8 = \lambda_8 - 1$, and $\hat{B} = B$. Using the transformation properties of these matrices under $SO(2) \times SO(3)$ and comparing with the $U(1)_R \times SU(2)_R$ charges of w and l_{jklm} , one learns for the Type IIB superstring that

$$w = \hat{U}, \quad l_{++++} = \hat{T}, \quad l_{+++-} = \hat{B}, \quad l_{+-} = \hat{\lambda}_8 \quad (4.8)$$

where the $SU(2)_R$ transformations are defined by commuting M_2 with

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that in the Type IIB (Type IIA) superstring, the lowest component of the chiral/chiral spacetime superfield W describes the chiral/chiral (chiral/anti-chiral) worldsheet modulus of T^2 , while the lowest component in the chiral/anti-chiral superfield L_{++++} describes the chiral/anti-chiral (chiral/chiral) worldsheet modulus of T^2 .

So near $\hat{U} = \hat{T} = \hat{B} = \hat{\lambda}_8 = 0$, the R^4 terms in the effective action of the Type IIB superstring appear as

$$\int d^8x [R_{++}^4 (\frac{\partial}{\partial \hat{U}})^4 f_V(\hat{U}) + \quad (4.9)$$

$$R_{+-}^4 \left((\frac{\partial}{\partial \hat{T}})^4 g(\hat{T}) + 24h\hat{\lambda}_8 + C(\hat{T}, \hat{\lambda}_8) \right) + \text{c.c.}]$$

where $C(\hat{T}, \hat{\lambda}_8)$ comes from non-perturbative contributions. The terms proportional to f_V and g come from one-loop since they are independent of λ_8 , while the loop dependence of the term proportional to h can not be determined from a linearized analysis. However, since there is precisely one λ_8 -dependent perturbative term at linearized level, there should be precisely one λ_8 -dependent term in the full non-linear action.

Therefore, in the full non-linear effective action of the Type IIB superstring on T^2 , the R^4 terms appear as

$$\int d^8x \sqrt{g_8} [R_{++}^4 A(U) + R_{+-}^4 (B(T) + h_g(\lambda_8)^{2g-2} + C(T, \lambda_8)) + \text{c.c.}] \quad (4.10)$$

where h_g is a constant. For the Type IIA superstring, the T and U moduli exchange places.

Up to terms coming from the holomorphic anomaly, this is in precise agreement with the results of [14] where

$$A(U) = 4\pi \log \eta(U), \quad B(T) = 4\pi \log \eta(T), \quad h_0 = 2\zeta(3).$$

Note that A and B are the same function, which does not follow from T -duality. It is analogous to the Type IIA/Type IIB relation between f_T and g in the N=2 D=4 action which was mentioned in footnote 1.

To determine R^4 terms in the effective action of the uncompactified Type II superstring, one needs to take T_2 to infinity and keep terms which diverge linearly with T_2 ,

remembering that λ_8 scales like $T_2^{-\frac{1}{2}}$. For the Type IIB superstring, the only terms which survive from (4.10) are

$$\int d^{10}x \sqrt{g_{10}} R_{+-}^4 [h_0(\lambda_{10})^{-2} + \lim_{T_2 \rightarrow \infty} T_2^{-1} (B(T) + C(T, \lambda_{10} T_2^{-\frac{1}{2}}))] + c.c. , \quad (4.11)$$

so R^4 terms only get tree-level, one-loop and non-perturbative contributions. For the Type IIA superstring, the only terms which survive are

$$\int d^{10}x \sqrt{g_{10}} \left(R_{+-}^4 h_0(\lambda_{10})^{-2} + R_{++}^4 \lim_{T_2 \rightarrow \infty} T_2^{-1} A(T) \right) + c.c. ,$$

so R^4 terms only get tree-level and one-loop contributions.

5. Possible Generalizations of the Non-Renormalization Theorems

In this paper, R^4 terms in the effective action of the Type II superstring compactified on T^2 were constructed in N=2 D=8 superspace, which allowed a simple proof of R^4 non-renormalization theorems. This superspace construction was very similar to the construction of the vector and hypermultiplet potentials in the N=2 D=4 superspace effective action of the D=4 Type II superstring.

As is well-known, there are higher-derivative topological amplitudes[11] of the D=4 Type II superstring which have properties similar to those of the vector and hypermultiplet potentials. These topological amplitudes come from terms which can be written in N=2 D=4 superspace as

$$\begin{aligned} \int d^4x |_{\theta_j^\alpha = \bar{\theta}_j^{\dot{\alpha}} = 0} [(D_+)^2 (D_-)^2 \left((P_{\alpha\beta} P^{\alpha\beta})^g A_g(W^{(I)}) \right) + \\ (D_-)^2 (\bar{D}^+)^2 \left((Q_{\alpha\dot{\beta}} Q^{\alpha\dot{\beta}})^g B_g(L_{++}^{(J)}) \right) + c.c.] \end{aligned} \quad (5.1)$$

where $P_{\alpha\beta}$ and $Q_{\alpha\dot{\beta}}$ are chiral and twisted-chiral field-strengths constructed from the supergravity multiplet. In components, these terms are

$$\int d^4x \sqrt{g_4} [R^2 (F_{\mu\nu} F^{\mu\nu})^{g-1} A_g(w^{(I)}) + R^2 (\partial_\mu Z \partial^\mu Z)^{g-1} B_g(l_{++}^{(J)}) + c.c.] + \dots$$

where $F^{\mu\nu}$ is the graviphoton field-strength and Z is a complex Ramond-Ramond scalar. Using arguments similar to those of section 2, one can prove that A_g only appears at genus g and B_g only appears at genus g and non-perturbatively.[4]

It is less well-known that there are also higher-derivative topological amplitudes of the D=8 Type II superstring which have properties similar to those of the R^4 term. These amplitudes were first shown to be topological for the Type II superstring compactified on $K3$ [15], and were later computed explicitly[16] for the Type II superstring compactified on $T^2 \times R^2$ (i.e. for the Type II superstring compactified on T^2).

For the Type IIA superstring, topological amplitudes at genus g come from terms of the form

$$\int d^8x \sqrt{g_8} [R^4(F^4)^{g-1} A_g(T, \bar{T}) + R^4(H^4)^{g-1} B_g(U, \bar{U}) + c.c.] + \dots \quad (5.2)$$

where F^4 is constructed from the vector field-strengths and H^4 is constructed from the tensor field-strengths. In the language of [16], this amplitude comes from the top instanton number and

$$A_g(T, \bar{T}) = (\lambda_8)^{2g-2} (T_2)^g \sum_{(m,n) \neq (0,0)} \frac{|n + mT|^{2g-4}}{(n + mT)^{4g-4}}, \quad (5.3)$$

$$B_g(U, \bar{U}) = (\lambda_8)^{2g-2} (U_2)^g \sum_{(m,n) \neq (0,0)} \frac{|n + mU|^{2g-4}}{(n + mU)^{4g-4}}.$$

For the Type IIB superstring, T and U exchange places.

As was first pointed out by Vafa[17], this suggests there might be non-renormalization theorems for $R^4 F^{4g-4}$ and $R^4 H^{4g-4}$ terms in the Type II superstring effective action. Note that these terms come from dimensional reduction of R^{2g+2} terms in eleven dimensions. At this moment, it is not known how to write the action of (5.2) in N=2 D=8 superspace, so the methods of this paper can not be used to check the existence of these non-renormalization theorems.

One mysterious feature of such a theorem is that it would naively imply that R^{2g+2} , like R^4 , receives contributions only at genus g and below in the effective action of the uncompactified Type IIA superstring. Note that A_g scales like T_2 as $T_2 \rightarrow \infty$, so this term appears to be present in ten dimensions, and to blow up like $(R_{11})^{g-1}$ in eleven dimensions. As pointed out in [18], lack of non-perturbative corrections to R^{2g+2} terms would violate eleven-dimensional covariance of M -theory since only R^{3g+1} terms can come from dimensional reduction of Lorentz-covariant terms in eleven dimensions.

Note that if R^{2g+2} were a topological term in the effective action of the uncompactified Type II superstring action, it would be reasonable for F^{2g+2} to be a topological term in the

effective action of the Type I superstring, since the photon vertex operator is the ‘square-root’ of the graviton vertex operator. The topological nature of such an F^{2g+2} term is supported by recent M(atrix) model computations[19] [20].

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